

A Simple Model for a Superfluid Stirling Refrigerator at High Operating Temperature

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The superfluid Stirling refrigerator (SSR) is a Stirling machine that uses the ideal gas behavior of the ^3He Component of a ^3He - ^4He mixture to provide cooling at temperatures below 2 K. The design of SSR's to date has used the Schmidt model to predict the performance of a given machine. Unfortunately, the Schmidt model does not account for the phonon and roton excitations present in ^4He component of the mixture. These excitations significantly change the performance of the SSR. An analytical model of the SSR that accounts for the phonon and roton excitations is presented. High temperature SSR experimental data is also presented and found to compare well with the model.

INTRODUCTION

Superfluid Stirling refrigerators (SSR) provide cooling below 2 K, by exploiting the unusual properties of superfluid ^3He - ^4He mixtures.¹ To first approximation, the ^3He component of the mixture behaves as an ideal Boltzmann gas in an inert background of superfluid ^4He . By using super-leak bypassed pistons, the ^3He component of the mixture is driven through the Stirling cycle to provide cooling. SSR's have demonstrated cooling to 0.17 K² and have demonstrated cooling powers of 977 μW at 0.5 K.³

The zeroth order design of all Stirling machines is done with the highly idealized Schmidt model,⁴ which assumes an ideal Boltzmann gas behavior for the working fluid. This model is extremely useful as a benchmark for all Stirling machine designs and provides a framework for understanding the overall trends in the design space. Unfortunately, the ideal Boltzmann gas model, and consequently the Schmidt model, is appropriate only for a limited temperature and concentration range of ^3He in ^4He which, in turn, makes the Schmidt model of limited use for superfluid

Stirling refrigerator design. At high temperatures, the thermal excitations in the ^4He , the phonons and rotons, significantly alter the thermodynamic and kinematic behavior of the working fluid from that of an ideal Boltzmann gas; whereas at low temperatures, the ^4He excitations are absent but the ^3He component of the mixture behaves as a Fermi, not a Boltzmann, gas.

Brisson and Swift⁵ have shown that the presence of these phonon-roton excitations in the ^4He component of the mixture can significantly augment the cooling power of the SSR. However their analysis and data were limited to SSR operation at uniform temperature. Here, we will extend the results of Brisson and Swift to the non-uniform temperature operation of an SSR where the phonon-roton effects are incorporated into an Schmidt-like analytical model. We will not, however, address in detail the low-temperature Fermi behavior of the ^3He here.

We believe that the phonon-roton (PR) model presented here is the appropriate zeroth order model for superfluid Stirling refrigerators. Because the model is of closed form, it is easily programmed onto a spreadsheet for design evaluation. In addition, it predicts several effects not predicted by the Schmidt model, such as a non-zero ultimate temperature and an enhanced cooling power above 1 Kelvin.

The paper is divided into several sections. First, a simple discussion of the basic Stirling cycle is described and how it relates to actual superfluid Stirling refrigerator designs. This is followed by a review of the Schmidt model for Stirling machines that establishes notation and provides a framework for the subsequent description of the phonon-roton model. The experimental SSR and the experimental procedure are then described and the experimental results are compared to the Schmidt and PR models. The penultimate section of this paper describes other limits to SSR performance such as heat flush, Fermi, and phase separation effects. The paper ends with some closing comments.

Cycle Basics

The superfluid Stirling cycle can be understood using Fig. 1, which depicts the four steps in an articulated Stirling cycle. The liquid ^3He - ^4He mixture is held between a cold piston and hot piston, each bypassed by superleaks. The volume behind of each of these pistons is filled with liquid superfluid ^4He . The piston cylinders are connected by a regenerator which can be thought of as an array of high heat capacity tubes with poor thermal conductivity in the longitudinal direction and good thermal conductivity in the transverse direction. (In the ideal regenerator the heat capacity of the tubes is infinite, the thermal contact between the fluid and the tubes is perfect and there is no thermal conductance in the longitudinal direction.)

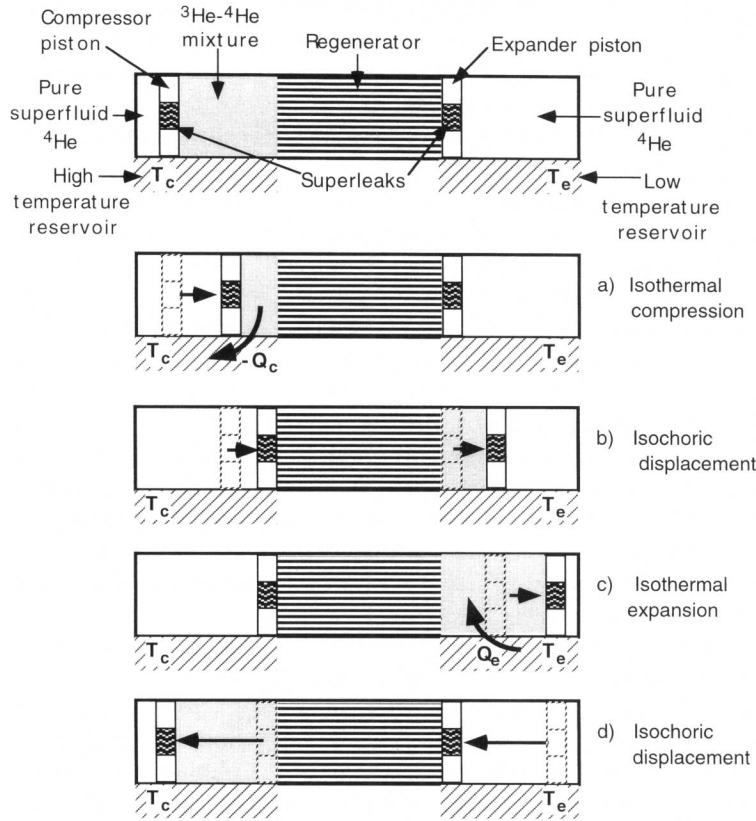


Fig. 1. A schematic diagram of the four stages of an articulated superfluid Stirling cycle. The compressor (expander) piston is assumed to be isothermal and in good thermal contact with a heat reservoir at high (low) temperature T_c (T_e).

We assume for the moment that the fluid volume contained in the regenerator is negligible when compared to the fluid in the piston cylinders.

The cycle begins with the ^3He - ^4He mixture entirely in the compressor piston cylinder; at this point, the mixture is at the same temperature as the high temperature reservoir, T_c . The normal ^3He component is isothermally compressed by the hot piston (Fig. 1a) while the superfluid ^4He component of the mixture flows freely through the superleak to the backside of the compressor piston. Because the ^3He component of the mixture behaves as an ideal gas in an inert "atmosphere" of ^4He , heat is rejected to the high temperature reservoir ($-Q_c$ in Fig. 1a). Following this, both pistons are slowly moved in tandem to isochorically displace the ^3He through the regenerator to the cold expander piston cylinder (Fig. 1b). Ideally, there is

a linear temperature gradient in the regenerator and, because of the large heat capacity of the regenerator matrix, the ^3He is reversibly cooled from the compressor piston temperature to the temperature of the cold thermal reservoir, T_c . The ^3He is now isothermally expanded, absorbing heat, Q_c , from the low temperature reservoir (Fig. 1c). Finally, the expanded ^3He is displaced back through the regenerator (where it reversibly reclaims the heat deposited there earlier) to the compressor piston cylinder (Fig. 1d). The refrigerator is back in the original state and the net effect has been a heat transfer from the cold reservoir to the hot reservoir.

A schematic diagram of a more practical SSR, first demonstrated by Brisson and Swift,⁶ is shown in Fig. 2. The refrigerator is actually two Stirling refrigerators operating 180° out of phase with each other. The compressor pistons are driven sinusoidally, 90° out of phase with the expander pistons. This back-to-back Stirling configuration allows the use of a recuperator (a counterflow heat exchanger) instead of a regenerator to provide the thermal isolation between the compressor and expander piston cylinders. The primary reason for a recuperative design over a regenerative design is the dearth of sub-Kelvin high-heat-capacity materials necessary for an efficient regenerator. This design also allows the removal of the pure ^4He space shown in Fig. 1 since each Stirling refrigerator half acts as a reservoir of ^4He for the other refrigerator.

The modeling of a recuperative Stirling machine (Fig. 2) can be reduced to a model of a regenerative Stirling machine (Fig. 1) by modeling

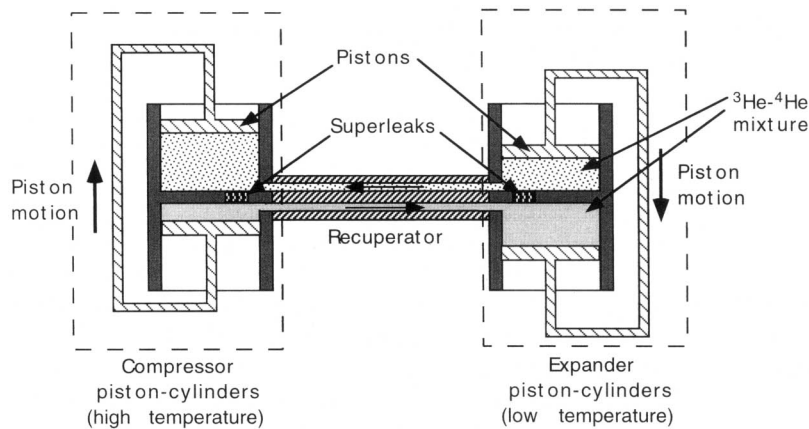


Fig. 2. A schematic diagram of a practical superfluid Stirling cycle. The mechanical linkage between the compressor pistons insures 180° phase shift between the compressor pistons. The expander pistons are similarly configured. The compressor and expander pistons are driven sinusoidally and 90° out of phase with each other. To first order, the flows in the recuperator are equal and opposite as shown.

each half of the recuperative Stirling machine as a regenerative machine. The flow on one side of the recuperator acts as an infinite heat capacity for the flow on the other side of the recuperator. So, from the perspective of the flow on one side of the recuperator, the recuperator looks like a regenerator. Since both the Schmidt model and the phonon-roton model, model only one side of the recuperative Stirling machine we will refer to the recuperator as the regenerator in these sections.

Schmidt Model of the SSR

We review here the Schmidt model for the superfluid Stirling refrigerator. This model is the zeroth order model for the design of any Stirling machine and was used extensively in the original designs of SSR's. The Schmidt model *is an ideal model; there are no viscous effects and all heat transfers occur reversibly*. Consequently, the efficiency of this cycle is that of Carnot. We outline the model here for completeness, to allow its comparison to the phonon-roton model, and to provide a simple framework for the understanding of the more complicated PR model. This treatment borrows liberally from Ureili's discussion of the Schmidt analysis of Stirling machines.⁷

Here, we analyze one side of the back to back superfluid Stirling refrigerators. This is entirely equivalent to modeling the SSR depicted in Fig. 1. The temperature of the liquid throughout each piston-cylinder is assumed to be at the temperature of the local heat reservoir. We also assume that the ^3He behaves as an ideal Boltzmann gas in an inert background of superfluid ^4He . The volumes of the piston cylinders are assumed to be driven sinusoidally, namely as

$$V_c = V_{cle} + V_{swe} \left(\frac{1 + \cos \theta}{2} \right) \quad (1)$$

$$V_e = V_{cle} + V_{swe} \left(\frac{1 + \cos(\theta + \alpha)}{2} \right) \quad (2)$$

where V_c , V_{cle} , V_{swe} are the compressor piston's total volume, clearance volume and swept volumes, respectively. Similarly, V_e , V_{cle} , V_{swe} are the expander piston's total volume, clearance volume and swept volume, respectively. The crank angle, θ , is assumed to vary linearly with time, $\theta = \omega t$, where ω is the angular frequency and t is time. There is a phase shift between the pistons, α , of 90° .

Because of the infinite heat capacity of the ideal regenerator matrix and, by assumption, the regenerator is in steady state, the temperature distribution in the ideal regenerator is independent of time. Since the mass

flow through the regenerator is balanced (the net mass flow through the regenerator is zero) and the heat capacity of the ideal gas is a constant, the temperature in the regenerator varies linearly with position from the compressor temperature to the expander temperature.

Since the pressure drops associated with the viscous flow of the ideal gas are assumed negligible, the pressure is spatially uniform throughout the Stirling machine but time varying. The total number of moles of ^3He , N_3 , in the Stirling refrigerator can be written as:

$$N_3 = \frac{pV_c}{RT_c} + \frac{pV_r}{RT_{LM}} + \frac{pV_e}{RT_e} \quad (3)$$

where p is the time varying pressure, R is the universal gas constant, V_r is the regenerator volume and, T_c and T_e are the compressor and expander operating temperatures. T_{LM} is the log mean temperature of the regenerator $(=(T_c - T_e)/\ln(T_c/T_e))$. From Eq. (3), the pressure associated with the ^3He is then

$$p = \frac{N_3 R}{\frac{V_c}{T_c} + \frac{V_r}{T_{LM}} + \frac{V_e}{T_e}}. \quad (4)$$

The time dependence of p is buried in the time dependent variation of V_c and V_e . The work per cycle done by the expander and compressor is determined by integrating $p dv$ for each piston over one cycle. The results are:

$$W_c = \pi V_{swc} \frac{NR}{s_s \sqrt{1-b_s^2}} \left(\frac{\sqrt{1-b_s^2}-1}{b_s} \right) \sin(\beta) \quad (5)$$

$$W_e = \pi V_{swe} \frac{NR}{s_s \sqrt{1-b_s^2}} \left(\frac{\sqrt{1-b_s^2}-1}{b_s} \right) \sin(\beta - \alpha) \quad (6)$$

where

$$s_s = \frac{V_{cle}}{T_c} + \frac{V_{swc}}{2T_c} + \frac{V_r}{T_{LM}} + \frac{V_{cle}}{T_e} + \frac{V_{swe}}{2T_e}, \quad (7)$$

$$\beta = \tan^{-1} \left(\frac{V_{swe} \sin \alpha / T_e}{V_{swe} \cos \alpha / T_e + V_{swc} / T_c} \right), \quad (8)$$

$$c = \frac{1}{2} \sqrt{\left(\frac{V_{swe}}{T_e} \right)^2 + \frac{2V_{swe}V_{swc}}{T_e T_c} \cos \alpha + \left(\frac{V_{swc}}{T_c} \right)^2}, \quad (9)$$

$$b_s = \frac{c}{s_s}. \quad (10)$$

The heat transfers to the gases in the expander can be calculated by applying energy conservation to the expander. For an ideal gas, the energy equation is

$$\frac{dE}{dt} = \dot{Q} - \dot{W} + \dot{N}_3 c_p T \quad (11)$$

where E is the internal energy of the ^3He in the control volume, $\dot{Q}$ and $\dot{W}$ are the rate of heat and work transfer, positive $\dot{N}_3$ is the molar flow rate of ^3He into the expander volume, c_p is the molar heat capacity at constant pressure ($=5/2R$) of the ^3He , and T is the temperature of the ^3He entering and exiting the expander. As we are dealing with ideal components, the temperature of the gas entering and leaving the expander is always T_e . If Eq. (11) is integrated over one cycle, the $\dot{N}_3 c_p T$ term reduces to a constant, $c_p T_e$, times a cyclic integral of the molar flow into and out of the control volume. This integral is zero by conservation of mass. Similarly, the internal energy term integrates to zero because the starting and finishing states are the same. We conclude that the heat absorbed by the expander is equal to the work done by the expander or $Q_e = W_e$. A similar analysis on the compressor results in the conclusion that the heat absorbed by the compressor is equal to the work done by the compressor or $Q_c = W_c$.

The coefficient of performance of the refrigerator, COP, is defined as the heat transferred to the expander divided by the total work absorbed by the refrigerator. Using Eqs. (5) and (6), we find the coefficient of performance of a Schmidt cycle refrigerator is

$$\text{COP} \equiv -\frac{Q_e}{W_c + W_e} = \frac{T_e}{T_c - T_e} \quad (12)$$

which is the Carnot coefficient of performance.

Phonon-Roton Model

Introduction

In the discussion of the Schmidt model for the SSR we blithely assumed that the contribution of the ^4He component of the mixture does not contribute to the performance of the SSR. Here, we account for the thermodynamic contribution of the normal component of the ^4He . Our goal is to determine the work and heat transfer in closed form for each of the SSR piston cylinders. To do this, we follow the same overall procedure described in the Schmidt model section. First, an expression for the pressure of the ^3He in the SSR is developed. The effect of the phonon-roton gas on the ^3He

concentration in the SSR is considered and an expression for the effective pressure on the SSR pistons is determined. This is compared to the corresponding expressions in the Schmidt model. Following this, the expressions for the work done on the pistons are determined by integrating the pressure-volume relations. To evaluate the heat transfer of the pistons, the first law is introduced. But, unlike the Schmidt model, there is non-zero contribution to the energy equation due to a net (osmotic) enthalpy flow through the regenerator. We then develop a simple model for the regenerator that allows the evaluation of the enthalpy term in the expander piston and subsequently the heat transfer in the expander piston-cylinder. The heat transfer in the compressor piston is then determined by applying the first law to the entire SSR. Finally, since the entropy and fountain pressure of pure ^4He are needed to evaluate the derived heat and work transfer expressions, we develop an empirical expression for these quantities by fitting to experimental data.

^3He Pressure in the SSR

If critical velocities are not exceeded, the superflows in the ^4He component will maintain a constant ^4He chemical potential throughout the SSR. Barring viscous effects, force balance requires that the pressure be uniform throughout the ^3He - ^4He mixture. Across the superleaks, the ^4He chemical potential is constant. If the mixture is in contact with a large reservoir of pure ^4He at pressure P_r and compressor temperature T_c , the ^3He component will distribute itself to satisfy the thermodynamic relation

$$\mu_4^0(P_r, T_c) = \mu_4(P_r + \Pi, T_c, x_c) = \mu_4(P_r + \Pi, T_e, x_e) \quad (13)$$

where μ_4^0 and μ_4 are the ^4He chemical potential in the pure state and in the ^3He - ^4He mixture, respectively, Π is the osmotic pressure, T_e is the temperature of the expander cylinder, and x_c (x_e) is the molar concentration of the ^3He in the compressor (expander) cylinder. In the actual SSR this large reservoir of pure ^4He is fictional; however, the effect of the opposing SSR is as if a large volume of pure ^4He exists at the center of the superleak between the two SSRs. The first equality of Eq. (13) is the defining relation for the osmotic pressure. The second equality relates the molar concentrations found in the compressor and expander cylinders for a given T_c and T_e .

We will assume here, contrary to the Schmidt model, that the regenerator volume is negligible. This simplifying assumption will be discussed in more detail later. The conservation of mass (moles) of ^3He in the space enclosed by the superleaks results in

$$N_3 = \int_c \frac{x}{v_m} dV + \int_e \frac{x}{v_m} dV = \left(\frac{x}{v_m} V \right)_c + \left(\frac{x}{v_m} V \right)_e \quad (14)$$

where N_3 is the total number of moles of ^3He in the refrigerator, x is the ^3He molar concentration, v_m is the molar volume of the mixture, and the subscripts c and e refer to evaluating the associated quantities with compressor or expander values. Equations (13) and (14) determine the ^3He concentrations in the refrigerator. To proceed further we require a relation for the ^4He chemical potential of the mixture. Kuerten and co-workers⁸ suggest that the ^4He chemical potential of a mixture can be written as

$$\mu_4(p, T, x) = \mu_4^0(p, T) - \Pi(T, x) v_4^0 \quad (15)$$

where v_4^0 is the molar volume of pure ^4He and p is the pressure of the mixture. The latter part of Eq. (13) can be expressed as the differential of μ_4 is zero throughout the ^3He - ^4He space or using Eq. (15):

$$d\mu_4^0 - v_4^0 d\Pi = v_4^0 dp - s_4^0 dT - v_4^0 d\Pi = -s_4^0 dT - v_4^0 d\Pi = 0 \quad (16)$$

where s_4^0 is the molar entropy of pure ^4He . The pressure term, $v_4^0 dp$, is zero since mechanical equilibrium requires that there are no pressure gradients in the ^3He - ^4He part of the device. The expression can now be integrated to

$$\int_0^T \rho_4^0 s_4^0 dT + \Pi(x, T) = \text{constant} \quad (17)$$

where ρ_4^0 is the molar density of pure ^4He . The first term is the fountain pressure of pure ^4He which we will label as p_f . The osmotic pressure is identified as the pressure of the quasiparticle gas $p_q = \Pi(x, T)$.⁹ Equation (17) reduces to

$$(p_f(T) + p_q(x, T))_c = (p_f(T) + p_q(x, T))_e. \quad (18)$$

Previous work on the SSR has always arranged to make the p_f terms in Eq. (18) insignificant. This is true at low temperatures for reasonable concentrations ($T < 1$ K, $x \sim 0.06$) because the fountain pressures are much smaller than the quasiparticle gas pressures. Watanabe *et al.*¹⁰ made high-temperature measurements that eliminate the mechanical fountain effects by running the SSR at uniform temperature since, in this case, the fountain terms cancel out identically.

Since our intent is to develop a simple Schmidt-like model for the high temperature operation of the SSR, we will assume that the ^3He quasiparticle gas behaves as an ideal Boltzmann gas, $p_q = n_3 RT$ where n_3 is the ^3He molar density in the mixture. The fountain pressure can be thought of as

the pressure of a phonon-roton gas of quasiparticles present in the ^4He component of the mixture. Equation (18) becomes

$$n_{3e}RT_e - n_{3c}RT_c = p_f(T_c) - p_f(T_e). \quad (19)$$

(The Schmidt model assumes that the right hand side of Eq. (19) is negligible and proceeds from this point.) If N_3 moles of ^3He are loaded into the SSR then Eq. (14) becomes

$$N_3 = n_{3c}V_c + n_{3e}V_e. \quad (20)$$

Equations (19) and (20), with the ideal gas relation for p_q , can be manipulated for the quasiparticle gas pressures in the compressor and the expander volumes, resulting in:

$$p_{qc} = \frac{N_3 R}{\frac{V_c}{T_c} + \frac{V_e}{T_e}} - \frac{(p_{fc} - p_{fe}) \frac{V_e}{T_e}}{\frac{V_c}{T_c} + \frac{V_e}{T_e}} \quad (21)$$

$$p_{qe} = \frac{N_3 R}{\frac{V_c}{T_c} + \frac{V_e}{T_e}} + \frac{(p_{fc} - p_{fe}) \frac{V_c}{T_c}}{\frac{V_c}{T_c} + \frac{V_e}{T_e}} \quad (22)$$

where $p_{fc} = p_f(T_c)$ and $p_{fe} = p_f(T_e)$. The first terms in these equations would be the result if the fountain pressures were ignored and corresponds to the Schmidt pressure of Eq. (4). In each of these equations, the second term is a modification in the pressure due to the fountain effect flushing ^3He atoms from the compressor piston cylinder into the expander piston cylinder (assuming $T_c > T_e$). Since the piston motions for the PR model are identical to the Schmidt model, Eqs. (1) and (2) can be substituted for the volumes in the numerators of Eqs. (21) and (22) resulting in

$$p_{qc} = \frac{N_3 R - \frac{\Delta P_f}{T_e} \left(V_{cle} + \frac{V_{swe}}{2} \right)}{\frac{V_c}{T_c} + \frac{V_e}{T_e}} - \frac{\Delta P_f V_{swe} \cos(\theta + \alpha)}{2T_e \left(\frac{V_c}{T_c} + \frac{V_e}{T_e} \right)} \quad (23)$$

$$p_{qe} = \frac{N_3 R + \frac{\Delta P_f}{T_c} \left(V_{cle} + \frac{V_{swc}}{2} \right)}{\frac{V_c}{T_c} + \frac{V_e}{T_e}} + \frac{\Delta P_f V_{swc} \cos(\theta)}{2T_c \left(\frac{V_c}{T_c} + \frac{V_e}{T_e} \right)} \quad (24)$$

where ΔP_f is the fountain pressure difference between the compressor and expander pistons ($\Delta P_f = p_{fc} - p_{fe}$). There is a time dependence in all the terms of Eqs. (23) and (24) (through θ , $V_c(\theta)$ and $V_e(\theta)$); however, the numerator of the first terms is a constant in time. These first terms are pressure oscillations that have a Schmidt character to them. The compressor piston has a reduced effective ^3He density whereas the expander piston has an enhanced effective ^3He density due to the fountain pressure. The second terms correspond to a modulated pressure oscillation directly related to the fountain pressure difference between the pistons. Equations (1) and (2) can be substituted for the volumes in Eqs. (23) and (24) resulting in

$$p_{qc} = \frac{(NR)_{eff,c}}{s(1+b\cos\phi)} - \frac{\Delta P_f V_{swe} \cos(\theta + \alpha)}{2T_e s(1+b\cos\phi)} \quad (25)$$

$$p_{qe} = \frac{(NR)_{eff,e}}{s(1+b\cos\phi)} - \frac{\Delta P_f V_{swc} \cos(\theta)}{2T_c s(1+b\cos\phi)} \quad (26)$$

where

$$(NR)_{eff,c} = N_3 R - \frac{\Delta P_f}{T_e} \left(V_{cle} + \frac{V_{swe}}{2} \right) \quad (27)$$

$$(NR)_{eff,e} = N_3 R + \frac{\Delta P_f}{T_c} \left(V_{cle} + \frac{V_{swc}}{2} \right) \quad (28)$$

$$s = \frac{V_{cle}}{T_c} + \frac{V_{swc}}{2T_c} + \frac{V_{cle}}{T_e} + \frac{V_{swe}}{2T_e} \quad (29)$$

$$\beta = \tan^{-1} \left(\frac{V_{swe} \sin \alpha / T_e}{V_{swe} \cos \alpha / T_e + V_{swc} / T_c} \right) \quad (30)$$

$$c = \frac{1}{2} \sqrt{\left(\frac{V_{swe}}{T_e} \right)^2 + \frac{2V_{swe} V_{swc}}{T_e T_c} \cos \alpha + \left(\frac{V_{swc}}{T_c} \right)^2} \quad (31)$$

$$\phi = \theta + \beta \quad (32)$$

$$b = c/s \quad (33)$$

The substitutions outlined in Eqs. (25)–(33) reduce the pressures oscillations to simple form where the time dependence is entirely contained in the arguments of the trigonometric functions through ϕ and θ . We have repeated the definitions of β and c for purposes of clarity.

Work Transfer to the Pistons

The work done by the gas can be calculated for each piston by directly integrating $p dv$ over the entire cycle. For the compressor piston, the work, W_c , is

$$W_c = - \int_0^{2\pi} \left(\frac{(NR)_{eff,c}}{s(1+b \cos \phi)} - \frac{\Delta P_f V_{swe} \cos(\theta + \alpha)}{2T_e s(1+b \cos \phi)} \right) \left(\frac{V_{swc} \sin \theta}{2} \right) d\theta. \quad (34)$$

Similarly, the expander piston work, W_e , is

$$W_e = - \int_0^{2\pi} \left(\frac{(NR)_{eff,e}}{s(1+b \cos \phi)} + \frac{\Delta P_f V_{swc} \cos(\theta)}{2T_c s(1+b \cos \phi)} \right) \left(\frac{V_{swe} \sin(\theta + \alpha)}{2} \right) d\theta. \quad (35)$$

The integrals can be evaluated by making a substitution of $y = \tan(\theta/2)$ and integrating by partial fractions. The results are:

$$\begin{aligned} W_c = & \frac{\pi V_{swc}}{s} \left(\frac{1}{\sqrt{1-b^2}} - 1 \right) \left(-\frac{(NR)_{eff,c}}{b} \sin(\beta) - \frac{\Delta P_f V_{swe}}{2T_e b^2} \sin(2\beta - \alpha) \right) \\ & + \frac{\pi V_{swc}}{s} \left(\frac{\Delta P_f V_{swe}}{2T_e \sqrt{1-b^2}} \cos(\beta) \sin(\beta - \alpha) \right) \end{aligned} \quad (36)$$

and

$$\begin{aligned} W_e = & \frac{\pi V_{swe}}{s} \left(\frac{1}{\sqrt{1-b^2}} - 1 \right) \left(-\frac{(NR)_{eff,e}}{b} \sin(\beta - \alpha) + \frac{\Delta P_f V_{swc}}{2T_e b^2} \sin(2\beta - \alpha) \right) \\ & - \frac{\pi V_{swe}}{s} \left(\frac{\Delta P_f V_{swc}}{2T_e \sqrt{1-b^2}} \sin(\beta) \cos(\beta - \alpha) \right) \end{aligned} \quad (37)$$

Conservation of Energy Applied to the Expander

We have determined the work done by the expander and the compressor pistons. In the case of the Schmidt model, we argued that these works are the same as the heat absorbed by each of the pistons. Unfortunately, this is not the case with the presence of the phonon-roton gas. Here, we evaluate the heat transfer experienced by the expander piston by applying the first law to the expander piston.

Ebner and Edwards¹¹ have shown that the conservation of energy equation for steady-flow no-shaft-work systems in helium mixtures can be

cast in terms of the osmotic enthalpy. Their arguments are easily extended to the unsteady-flow with shaft-work case. The result is

$$\frac{dE}{dt} = \dot{Q} - \dot{W} + \sum_{ports} \dot{N}_3 h_3^{os} + \sum_{ports} \dot{N}_4 \mu_4 \quad (38)$$

where E is the total energy in the control volume, $\dot{Q}$ is the rate of heat transfer into the control volume, $\dot{W}$ is the rate of work transfer out of the control volume, $\dot{N}_3$ is the ^3He molar flow rate into the control volume, h_3^{os} is the osmotic enthalpy per mole of ^3He , $\dot{N}_4$ is the molar flow rate of the ^4He into the control volume and μ_4 is the ^4He chemical potential. The indicated sums are over the ports that allow the mixture to flow into and out of the control volume. We have assumed in writing Eq. (38) that the gravitational and kinetic energies are not significant. Applying Eq. (38) to the expander piston cylinder and integrating over a complete cycle, the conservation of energy reduces to

$$Q_e = W_e - \oint h_3^{os} dN_3 \quad (39)$$

where Q_e is the heat absorbed per cycle by the expander, W_e is the work per cycle done by the expander. The dE/dt term in Eq. (38) does not contribute because the internal energy is the same at the beginning and end of the cycle. The $\sum \dot{N}_4 \mu_4$ term also does not contribute to the integral. The ^4He chemical potential is held constant by the large ^4He reservoir in contact with the SSR through the superleaks. Since the ^4He chemical potential is constant throughout the cycle, the cycle integral reduces to summing of all the ^4He molar flow into and out of the control volume over a complete cycle. This is zero by conservation of mass.

In order to find Q_e , it is necessary to evaluate the cyclic integral of the osmotic enthalpy, which is defined as¹¹

$$h_3^{os} \equiv \mu_3 + \frac{T s_m}{x} = \mu_q + (p - p_q) v_3 + \frac{T s_m}{x} = h_q + (p - p_q) v_3 + \frac{(1-x)}{x} T s_4^0 \quad (40)$$

where μ_3 is the ^3He chemical potential, T is the temperature, s_m is the molar entropy of the mixture, μ_q is the chemical potential of the ^3He quasiparticle gas, p is the pressure of the system, v_3 is the partial volume of the ^3He , x is the molar concentration of ^3He , h_q is the molar quasiparticle gas enthalpy. The cyclic osmotic-enthalpy integral in Eq. (39) reduces to three integrals: an integral of the ^3He quasiparticle gas enthalpy

($\oint h_g dN_3$), a flow work integral ($\oint (p - p_q) v dN_3$), and an entropy flow integral ($\oint ((1 - x)/x) Ts_4^0 dN_3$).

It is easily shown that the flow work integral has value zero. The partial volume of the ^3He is related to the concentration dependence of the molar volume of the ^3He - ^4He mixture. DeWaele¹² suggests that the molar volume of the mixture, v_m , can be written as:

$$v_m = v_4^0(1 + \alpha'x) \quad (41)$$

where v_4^0 is the molar volume of pure ^4He ($= 27.58 \times 10^{-6} \text{ m}^3/\text{mol}$) and $\alpha' = 0.286$. The partial volume is calculated using the partial volume relation¹³ for binary mixtures:

$$v_3 = v_m + (1 - x) \frac{dv_m}{dx} = v_4^0(1 + \alpha'). \quad (42)$$

We see then that the partial volume of ^3He is constant. We have assumed that the SSR is connected to a large volume of ^4He at constant pressure, P_r , and temperature, T_r . As suggested by Eq. (13), the difference $p - p_q$ is equal to the constant P_r . The cycle integral in Eq. (39) of the $(p - p_q) v_3$ term reduces to a constant times the cycle integral of the number of ^3He particles in the volume and by conservation of mass, the flow work integral is zero.

The other two integrals cannot be dispatched with the same ease as the flow work integral because these integrals depend on the temperature of the fluid entering and leaving the expander. In order to evaluate these integrals, we will find in the next section that the temperature of the fluids entering and exiting the expander is at the time invariant temperature of the expander's thermal reservoir (T_e).

The Regenerator

In modeling the regenerator in the above cycle we have assumed that the regenerator has zero volume and is 100% effective. This is actually more restrictive than the standard Schmidt model, which allows a non-zero volume in the regenerator and compensates for this volume in the conservation of mass equation by using a log-mean temperature for the regenerator temperature. Unfortunately, due to the thermal imbalance introduced by the phonons and rotons present in the fluid mixture, the temperature distribution in the recuperator of the SSR is spatially non-linear. A thorough treatment of the regenerator is beyond the scope of this

work; however, several simplifying assumptions can be used in our phonon-roton model.

Due to the presence of the phonon-roton gas in the mixture (the normal ^4He component), the heat capacity of the mixture per mole of ^3He is larger for a low ^3He concentration mixture than it is for a high ^3He concentration mixture.¹⁴ In the superfluid Stirling cycle, the low concentration mixture (with high heat capacity) flows from the low temperature expander to the high temperature compressor and the high concentration mixture (with low heat capacity) flows from the high temperature compressor to the low temperature expander. There is a net cooling effect throughout the regenerator due to the imbalance of these heat capacity flows. The temperature distribution in the regenerator is no longer linear. In the extreme case where there is no longitudinal heat conduction and the heat transfer coefficient between the regenerator and fluid is infinite, the temperature profile in the regenerator is uniformly at the expander temperature. Only differentially close to the compressor end of the regenerator does the regenerator temperature approach the compressor temperature. We conclude then that the temperature of the fluid entering and exiting the expander end of the ideal regenerator is always at the expander temperature.

The corresponding comment for the compressor end of the regenerator is not true. There is a significant temperature difference between the fluid entering and exiting the compressor due to the large imbalance in the heat capacities. In the PR model, it is the cold fluid stream exiting the regenerator into the hot compressor piston and the hot fluid stream exiting the hot compressor piston into the cold regenerator that produces the irreversibilities of the cycle.

The imbalance in the heat capacities can be understood by considering the normal part of the ^4He component of the mixture, the phonons and the rotons, that are swept along with the ^3He .¹⁵ The phonon-roton gas volumetric density is a function only of the temperature and not the ^3He concentration. When ^3He flows from the compressor to the expander at high concentration, the ^3He sweeps a fairly small volume of phonons and rotons through the recuperator and therefore, relatively few phonons and rotons need be annihilated to cool the mixture from the compressor to expander temperature. On the other hand, when the ^3He flows from the expander to the compressor at low concentration, a large volume of phonons and rotons are swept through the recuperator and therefore, many phonons and rotons are created in warming the mixture from the expander to the compressor temperature. Since the energy and entropy absorbed (rejected) by the ^4He component of the mixture is proportional to the number of created (annihilated) phonons and rotons, we see then

that the low concentration flow has a higher heat capacity per ^3He atom than the high concentration flow.

In a real system, the temperature distributions in the regenerator are smoother than described in the ideal case due to imperfect transverse conduction to the regenerator walls and to longitudinal conduction.

Cooling Power

With the temperature of the flows entering and exiting the expander piston determined to be T_e , the two remaining cyclic integrals can be calculated and the cooling power of the phonon-roton model can be evaluated. In the ideal gas model, the enthalpy of the ^3He quasiparticle gas, h_q , is just $5/2RT$. Incorporating the observations of the past few sections into Eq. (39) we find

$$Q_e = W_e - \frac{5}{2} RT_e \oint dN_3 - T_e s_4^0 \oint \frac{1-x}{x} dN_3 = W_e - T_e s_4^0 \oint \frac{1}{x} dN_3. \quad (43)$$

The second equality is true because $\oint dN_3 = 0$ by conservation of mass. The last integral in Eq. (43) can be expanded as

$$\begin{aligned} \oint \frac{1}{x} dN_3 &= \oint \frac{(V_e dn_3 + n_3 dV_e)}{n_3 v_m} = \oint \left(\frac{1}{n_3 v_4^0} - \alpha' \right) (V_e dn_3 + n_3 dV_e) \\ &= \oint \frac{dV_e}{v_4^0} + \oint \frac{V_e}{v_4^0} \frac{dn_3}{n_3} - \alpha' \left(\oint n_3 dV_e + \oint V_e dn_3 \right) \end{aligned} \quad (44)$$

In the final expression of Eq. (44), the first integral is zero because the final volume is the same as the initial volume. The parenthetical sum multiplying α' is zero by conservation of the ^3He mass. Using these results Eq. (43) becomes

$$Q_e = W_e - \frac{T s_4^0}{v_4^0} \oint \frac{V_e dn_3}{n_3}. \quad (45)$$

The cyclic integral simplifies to

$$\begin{aligned} \oint \frac{V_e dn_3}{n_3} &= \pi b V_{swe} \sin(\beta - \alpha) \left(\frac{1}{\sqrt{1-b^2}} - \frac{1}{b^2} \left(\frac{1}{\sqrt{1-b^2}} - 1 \right) \right) \\ &\quad + \pi d V_{swe} \sin(\alpha) \left(\frac{1}{\sqrt{1-d^2}} - \frac{1}{d^2} \left(\frac{1}{\sqrt{1-d^2}} - 1 \right) \right) \end{aligned} \quad (46)$$

where

$$d = \frac{\Delta P_f V_{swc}}{2T_c(NR)_{eff,e}}.$$

The total cooling power of the refrigerator is easily calculated by substituting Eqs. (37) and (46) into Eq. (45).

Heat Transfer in the Compressor

The heat transfer experienced by the compressor is most easily evaluated by applying the conservation of energy to the entire SSR,

$$Q_c = -Q_e + W_c + W_e \quad (47)$$

where $-Q_c$ is the heat rejected by the SSR compressor to the environment. All the quantities on the right hand sides of the Eq. (47) are easily determined using Eqs. (36), (37), (45) and (46). The COP of the refrigerator is determined using the definition contained in Eq. (12). Finally, we define a second law efficiency as the ratio of the refrigerator COP to that of a reversible refrigerator operating between the same two temperatures:

$$\eta_2 = \frac{\text{COP}}{\text{COP}_{rev}} = \text{COP} \frac{T_c - T_e}{T_e}. \quad (48)$$

Entropy and Fountain Pressures of Pure ^4He

We have now generated expressions for the compressor and expander works and heat transfers. These expressions depend on the volumes and temperatures of the compressor and expander, the number of moles of ^3He , the fountain pressure difference between the expander and compressor, and the entropy of pure ^4He at the expander temperature (s_4^0 in Eq. (45)). Once again, we are interested in developing a closed form expressions for the work and heat transfers of the SSR. To this end then, it is necessary to have closed form expressions for the entropy and fountain pressures of pure ^4He .

We have fit the entropy as a function of temperature to published entropy values of pure ^4He ¹⁶ with an equation of the form:

$$s_4^0 = 4BT^3 + A \left(\frac{\Delta_1}{T} + \frac{3}{2} \right) \sqrt{T} e^{-\Delta_1/T} + C \left(\frac{\Delta_2}{T^2} \right) e^{-\Delta_2/T}. \quad (49)$$

The fit parameters A , B , C , Δ_1 and Δ_2 have values of $23.2 \text{ J/mole K}^{3/2}$, $6.75 \times 10^{-3} \text{ J/mole K}^4$, 500 J/mole , 8.65 K and 15.7 K , respectively. This expression is a modified form of the entropy expression given by Khalatnikov¹⁷ for the phonon-roton gas present in pure ^4He . The first term is the contribution of phonons to the ^4He entropy. The second term is the contribution of the rotons to the entropy of the ^4He , Δ_1 is the characteristic energy associated with a roton excitation. The last term can be thought of as the contribution of the "maxons" to the entropy of the gas and is necessary to improve the high temperature, $1.2 \text{ K} < T < 2 \text{ K}$, fit of the data. A graph of the fit is shown in Fig. 3. The relative error between the Eq. (49) and the values given by Wilkes is typically less than 4% and never exceeds 8%.

The data used in determining the coefficients used in Eq. (49) are entropies along the saturated vapor curve. We have assumed that the entropy is not a function of pressure throughout the range of interest. For liquid ^4He at temperatures between 1 and 2 K, a pressure change from one to two atmospheres induces a change in the entropy of the fluid of less than 1%.¹⁸ The pressure terms that were neglected in formulating Eq. (49) are smaller than the error associated with the fit. The entropy-independence-of-pressure assumption is also consistent with our analysis of the SSR where

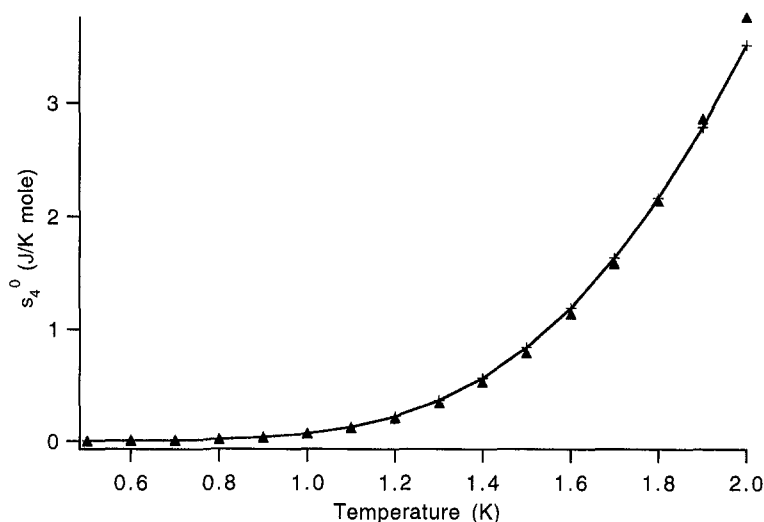


Fig. 3. A plot of the molar entropy of pure ^4He as a function of temperature. The dark triangles are experimentally derived values taken from a table of molar entropy in Wilkes. The crosses connected by lines are the result of the fit of Eq. (49) to these triangle points.

we have assumed that the helium mixture behaves as an incompressible fluid.

The fountain pressure of pure ^4He is linked to the entropy relation by London's expression:

$$p_f(T) \equiv \int_0^T \rho_4^0 s_4^0 dT \quad (50)$$

which was used in Eqs. (17) and (18) as the definition of fountain pressure. Using Eqs. (49) and (50) and assuming the density of the liquid is independent of temperature, the fountain pressure is

$$p_f(T) = \rho_4^0 (BT^4 + AT^{3/2}e^{(-\Delta_1/T)} + Ce^{(-\Delta_2/T)}). \quad (51)$$

With these expressions for the fountain pressure and the entropy, we now have a set of analytic expressions for the work and heat transfers of an SSR.

Experimental Apparatus

The experimental apparatus used in this work is similar to that discussed in detail in Patel and Brisson¹⁹ so that we will only briefly describe the apparatus here. A drawing of the apparatus is shown in Fig. 4. The compressor is cooled using a ^4He evaporation refrigerator (1 K pot) which is not shown in the figure.

This SSR is of the Brisson and Swift type having two SSR's operating 180° out of phase with each other and using a recuperator. The compressor and expander piston platforms of this SSR are solid blocks of OFHC copper on which pistons of edge welded stainless steel bellows are mounted. The effective areas and strokes of the compressor pistons are 17.74 cm^2 and 1.0 cm , respectively, and for the expander pistons 13.61 cm^2 and 0.98 cm . Within each piston platform, there are isothermal heat exchangers made from nested OFHC copper cylinders press fit into the piston platforms. A $76 \mu\text{m}$ gap exists between the inner wall of an outer cylinder and the outer wall of the inner cylinder. In total, there is 65.94 cm^2 of heat transfer area in the compressor piston platform and 209.70 cm^2 of heat transfer area in the expander piston platform. The recuperator of this SSR is made from Kapton and consists of two-hundred 2.38 mm by $127 \mu\text{m}$ by 20 cm long rectangular passages arranged as a counterflow heat exchanger and separated by a $25 \mu\text{m}$ Kapton septum. The total volume of the recuperator is 20.18 cm^3 of which 12.10 cm^3 (6.05 cm^3 per side) is devoted to recuperative heat transfer. The total volume of the SSR is 87.2 cm^3 (43.6 cm^3 per side).

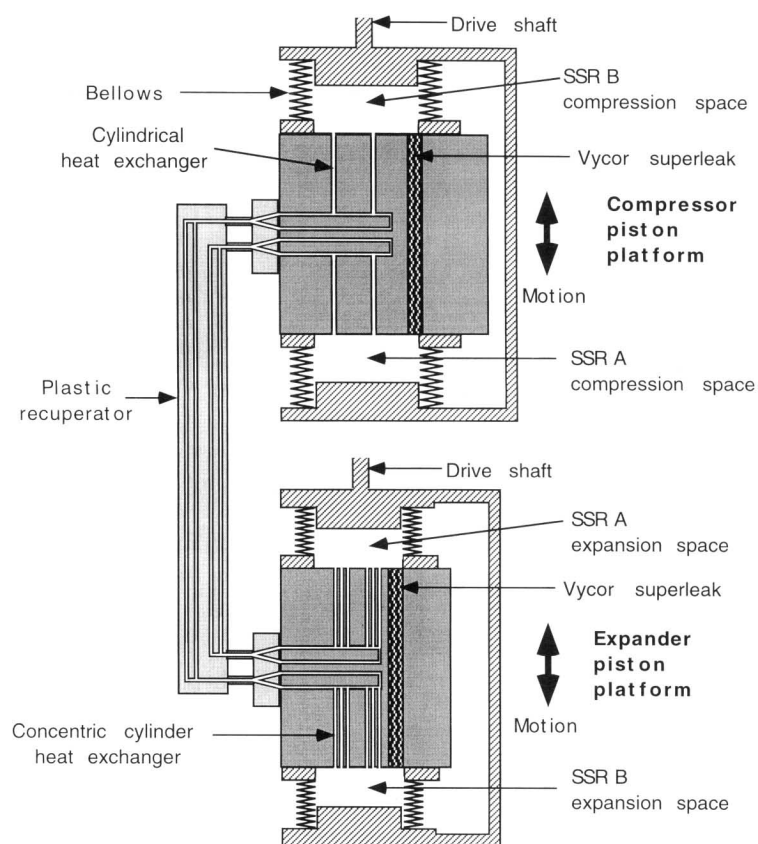


Fig. 4. A drawing of the superfluid Stirling refrigerator used in this investigation. The pistons are made with bellows to avoid the leaks and the friction associated with sliding seals. The recuperator is a plastic composite and the isothermal heat exchangers are made from press fit concentric copper cylinders. A pumped ^4He refrigerator (not shown) provides cooling for the compressor pistons.

The SSR was operated with a cycle period of 20 seconds and with a 3% ^3He - ^4He mixture. Data was taken for several compressor temperatures. The gross control of the compressor temperature was set by throttling the ^4He flow from the 1 K pot. The fine temperature control was done using a feedback controlled electric heater on the compressor platform. Fixed heat loads were placed on the expander platform using an electric heater. For each heat load, the expander platform temperature was allowed to come to steady state and then the expander platform temperature was recorded. The compressor and expander temperatures were determined using calibrated ruthenium oxide resistors.²⁰

Model versus Experiment

There is some ambiguity as to what volume values should be chosen to model our experimental SSR. The superleaks themselves are 28% void space and hence can contain significant amounts of mixture. However, we assume that the ^3He component that has diffused into the superleak is immobile on the time scale of a single Stirling oscillation. We also assume that the ^3He - ^4He mixture concentration in the Vycor superleak is the same as that of the bulk mixture in the SSR. Under these assumptions, the effect of the superleak on the SSR performance can be neglected. Practically speaking, the Vycor superleaks require the loading of some extra mixture into the SSR to fill them but they do not alter the concentration or the dynamics of the ^3He component of the mixture.

For one side of the SSR, the compressor clearance and swept volumes are 12.04 and 17.74 cm³, respectively. Similarly, the expander clearance and swept volumes are 9.83 and 13.34 cm³, respectively. The clearance volumes include the isothermal heat exchanger volumes and any “connecting” (headering) volumes between the piston, the isothermal heat exchanger and the recuperator heat exchange surfaces. The volume associated with the recuperator heat exchange surfaces (6.05 cm³) can be distributed between the clearance volumes of the compressor and expander pistons. In the calculations that follow, we have arbitrarily added half of this volume to the compressor clearance volume and half to the expander clearance volume.

All the calculated cooling powers and works reported in this article are double what would be calculated from the Schmidt and PR model expressions. This was done to account for the two back-to-back SSR's used in the experiment.

Comparison of the PR Model to the Schmidt Model and Experiment

Figure 5 shows plots of experimental cooling powers, Schmidt model cooling power and the PR model cooling power for the superfluid Stirling refrigerator for a compressor operating temperature of 1.2 K. In this case, the fountain pressure is significantly smaller than the osmotic pressure associated with the ^3He component of the mixture so that the influence of the phonon-roton gas on the motion of the ^3He gas is minimal. The reduced cooling power of the PR model compared to that of the Schmidt model at temperatures below 0.8 K is primarily due to the reduced mass flow of ^3He between the compressor and expander stages in the PR model. (The phonon-roton gas in the compressor piston-cylinder reduces the density of the ^3He in the compressor and increases the density in the expander piston.)

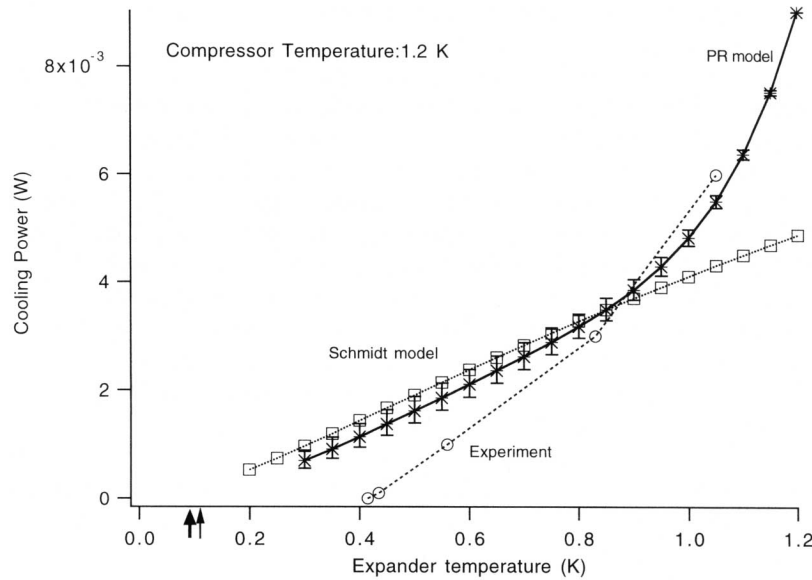


Fig. 5. A plot of the SSR cooling power versus expander temperature. The compressor is operated at 1.2 K. The three traces correspond to experiment (circles), the Schmidt model (squares) and PR model (*). A 3% ^3He in ^4He mixture is assumed.

The real performance of Stirling refrigerators is always significantly worse than the performance predicted by the Schmidt model, since this model does not consider any loss mechanisms. The high temperature performance of the SSR in Fig. 5 is a rare example of a real Stirling machine outperforming the predictions of the Schmidt model and is a clear indicator that the ideal gas model is an inadequate representation of the high temperature behavior of the working fluid. The large increase seen in both the experimental and the PR model cooling power curves at high temperature comes from the additional entropy flow due to the phonon-roton gas. When the expander is at high temperature ($T_e > 0.9$ K), significant numbers of phonon and rotons are generated in the expansion process. These phonons and rotons are subsequently swept up the recuperator with the ^3He quasiparticle gas to the compressor. These phonons and rotons are then destroyed in the compression process, releasing their energy and entropy to the high temperature heat exchangers.

It should be noted that the heightened cooling power of the PR model over that of the Schmidt model does not constitute a violation of the second law because there is a net increase in the work required to drive the SSR. We will show that the presence of the phonon-roton gas serves at

high temperature to increase the cooling power of the SSR at a cost of an increase in entropy production.

We attribute the large differences at low temperature between the experiment and the PR model to several effects. The first is that the thermal conductance (the UA product) between the fluid streams in the recuperator is not infinite. There are losses in the real machine due to the ineffectiveness of the heat exchangers. We would expect these losses to scale with the difference of the compressor and expander temperatures. The experimental data agrees well with the PR model when this temperature difference is small ($0.8 \text{ K} < T_c < 1.2 \text{ K}$ in Fig. 5). Second, due to Fermi effects, the equation of state of the ^3He component of the mixture will deviate from the Boltzmann ideal gas law at low temperature. Third, the actual expansions and compressions in the piston-cylinders are not entirely isothermal as assumed in the PR model.

The error bars shown on the PR curve in Fig. 5 (and in Figs. 6–8) correspond to the variation of the calculated cooling power when the recuperator volume is incorporated entirely into the compressor clearance volume in one extreme and entirely into the expander clearance volume in the other extreme.

Figure 6 consists of plots of experimental cooling power, Schmidt model cooling power and the PR model cooling power for the superfluid

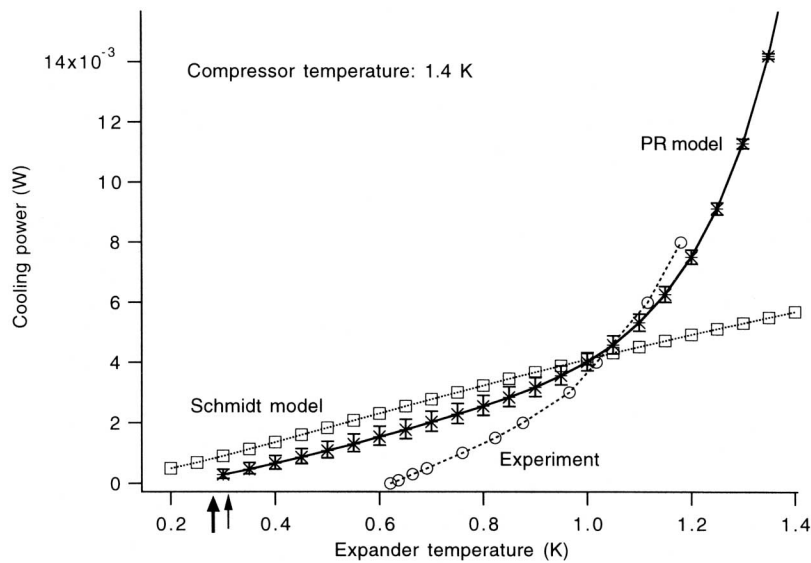


Fig. 6. A plot of the SSR cooling power versus expander temperature. The compressor is operated at 1.4 K. The three traces correspond to experiment (circles), the Schmidt model (squares) and PR model (*). A 3% ^3He in ^4He mixture is assumed.

Stirling refrigerator for a compressor operating temperature of 1.4 K. As expected, the Schmidt model cooling power is linear with the expander temperature. The PR model predicts a lower cooling power than the Schmidt model due to the flushing of some of the ^3He out of the high temperature piston and the reduction of the ^3He mass flow through the recuperator. Notice that this effect is more pronounced with a compressor temperature of 1.4 K (Fig. 6) than with a compressor temperature of 1.2 K (Fig. 5). Contrary to the Schmidt model, the PR model predicts that there is a temperature at which the cooling power of the refrigerator is zero. Extrapolating the PR trace in Fig. 6 results in an estimated minimum operating temperature for the refrigerator is below 0.2 K as opposed to the experimental data that shows a minimum temperature of 0.621 K. We will discuss the minimum operating temperature of the SSR in more depth shortly.

Figure 7 consists of plots of experimental cooling power, Schmidt model cooling power, and the PR model cooling power for the superfluid Stirling refrigerator for a compressor operating temperature of 1.6 K. The

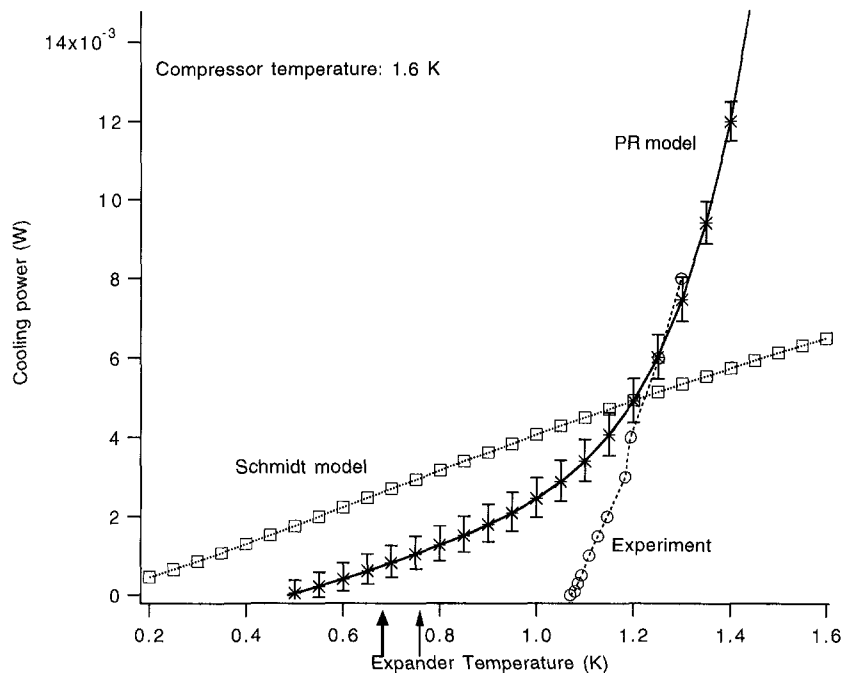


Fig. 7. A plot of the SSR cooling power versus expander temperature. The compressor is operated at 1.6 K. The three traces correspond to experiment (circles), the Schmidt model (squares) and PR model (*). A 3% ^3He in ^4He mixture is assumed.

deviation of the PR model from the Schmidt model is more pronounced than in the two earlier cases. The PR model minimum operating temperature has increased to 0.5 K versus the experimental minimum operating temperature of 1.07 K.

Finally, Fig. 8 shows plots of experimental cooling power, Schmidt model cooling power and the PR model cooling power for the superfluid Stirling refrigerator for a compressor operating temperature of 1.8 K. The PR model and experimental cooling power curves no longer resemble the Schmidt model trace. Both of the PR model and the experimental traces show a clear minimum operating temperature of 1.06 K and 1.46 K, respectively.

The expander and compressor works (W_e , W_c , respectively), cooling power (Q_e), waste heat (Q_c) and the second law efficiency (η_2) for the PR model of the experimental SSR above are shown in Fig. 9. The compressor temperature is assumed fixed at 1.8 K and the ^3He density corresponds to a 3% mixture.

One of the very clear differences between this model and the Schmidt model is the second law efficiency. In the Schmidt model, the second law efficiency has a constant value of one; however, in the PR model, the

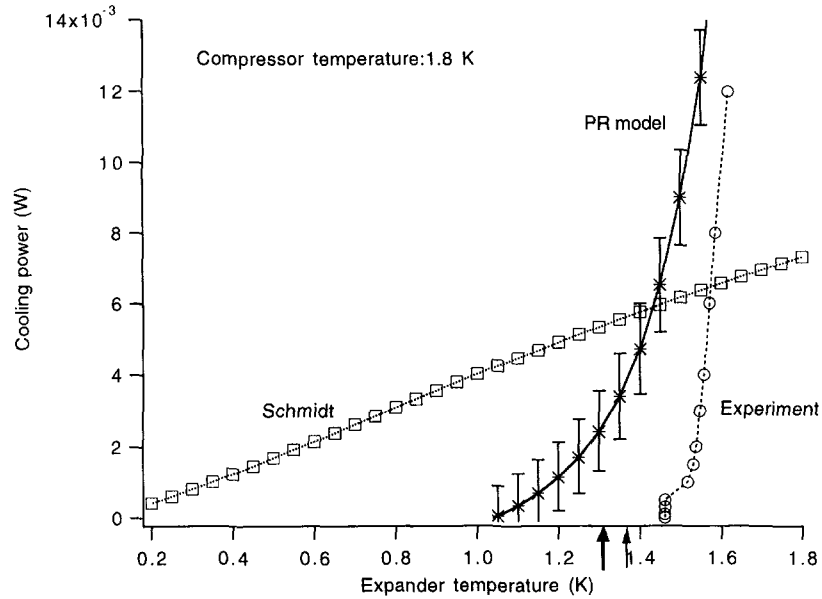


Fig. 8. A plot of the SSR cooling power versus expander temperature. The compressor is operated at 1.8 K. The three traces correspond to experiment (circles), the Schmidt model (squares) and PR model (*). A 3% ^3He in ^4He mixture is assumed.

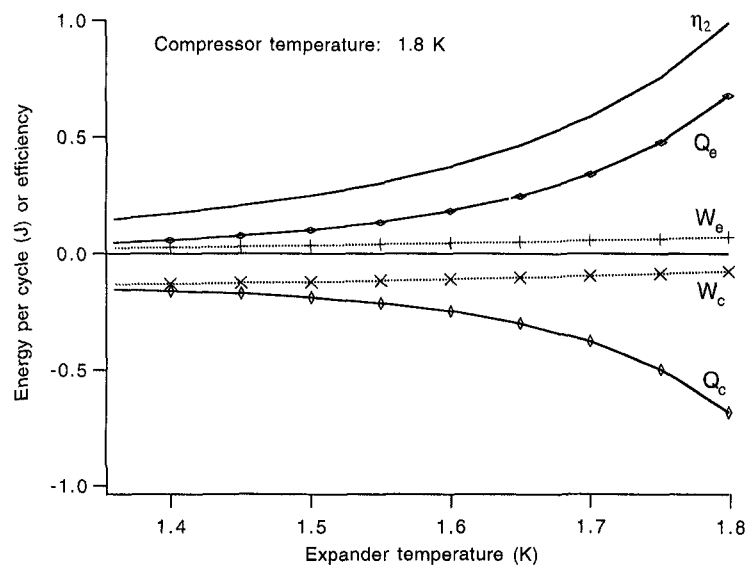


Fig. 9. Plots of the expander and compressor works (W_e and W_c , respectively), expander and compressor heats (Q_e and Q_c , respectively) and the second law efficiency, η_2 , as a function of expander temperature for the PR model. The compressor temperature is assumed to be 1.8 K and the loaded ^3He concentration is 3%.

second law efficiency drops to zero due to the heat capacity flow mismatch in the recuperator. Figure 10 shows the second law efficiency of the modeled SSR as a function of expander temperature for several compressor temperatures. On this graph, a Schmidt model second law efficiency would appear as a horizontal line with a value of $\eta_2 = 1$. As expected, in the low temperature compressor limit ($T_c = 0.8$ K), the PR model efficiency approaches that of the Schmidt model. The plots of efficiency become suspect at very low temperatures $T \leq 0.5$ K due to the change of the ^3He from Boltzmann gas behavior to Fermi gas behavior.

Ultimate Temperatures

The minimum expander temperature achievable by an SSR is a function of the compressor temperature, the average ^3He concentration and the ratio of the compressor and expander volumes. Practically speaking, the ultimate temperature is always influenced by the recuperator effectiveness and expander efficiency; but here, we are interested in determining an ultimate temperature based on 100% effective components.

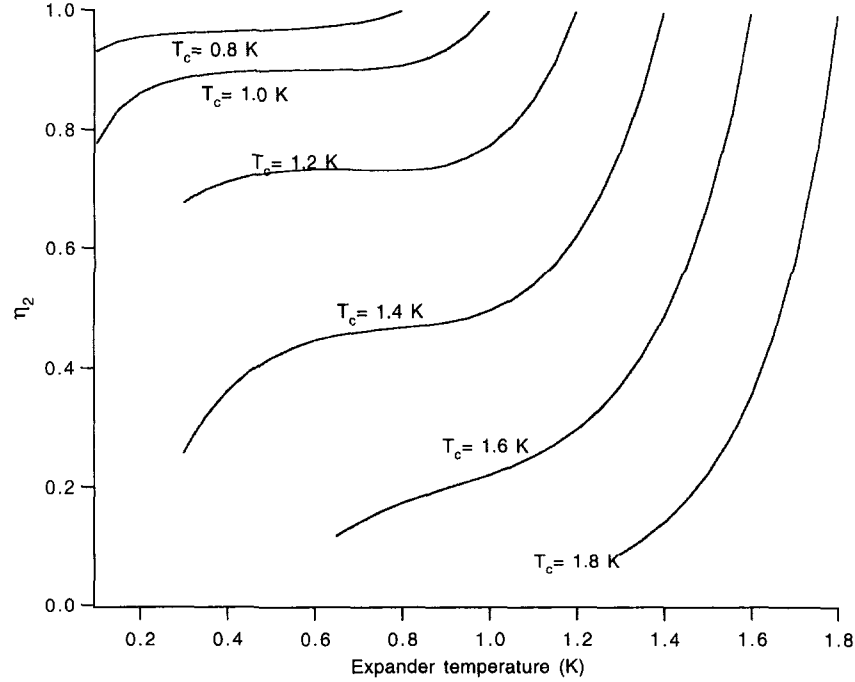


Fig. 10. Parametric curves for the second law efficiencies for the PR model as a function of the expander temperature. The volumes assumed correspond to the experimental apparatus.

The minimum achievable temperature is linked to the flushing of all the ^3He from the compressor piston into the expander piston due to the fountain effect. When this happens, there is a column of high-thermal-conductivity pure ^4He between the high temperature compressor and the low temperature expander which thermally “shorts” the refrigerator. The expander temperature at which this occurs can be determined by setting p_{qc} in Eq. (21) equal to zero. The result is a transcendental expression for $T_e|_{\min}$, the minimum expander operating temperature:

$$T_e|_{\min} = \frac{(p_f(T_c) - p_f(T_e|_{\min})) V_e|_{\max}}{N_3 R} \quad (52)$$

where $V_e|_{\max}$ is the maximum volume of the expander ($= V_{cle} + V_{swe}$). In general, the SSR is loaded with a mixture of known concentration when both the compressor and expander pistons are in their center positions and

at uniform temperature. This means that total number of moles of ^3He can be written as:

$$N_3 = n_{3, \text{load}} \left(V_{cle} + \frac{V_{swc}}{2} + V_{cle} + \frac{V_{swe}}{2} \right) = \frac{\left(V_{cle} + \frac{V_{swc}}{2} + V_{cle} + \frac{V_{swe}}{2} \right) x_{load}}{v_4^0 (1 + \alpha' x_{load})} \quad (53)$$

where $n_{3, \text{load}}$ and x_{load} are the molar ^3He density and the molar concentration of the mixture when it is loaded into the SSR. Substituting Eq. (53) into Eq. (52) we find:

$$\begin{aligned} T_e|_{\min} &= \left[\frac{V_{cle} + V_{swe}}{V_{cle} + (V_{swc}/2) + V_{cle} + (V_{swe}/2)} \frac{(1 + \alpha' x_{load})}{x_{load}} \right] \\ &\quad \times \frac{v_4^0 (p_f(T_c) - p_f(T_e|_{\min}))}{R} \\ &= \xi \frac{v_4^0 (p_f(T_c) - p_f(T_e|_{\min}))}{R} \end{aligned} \quad (54)$$

where the dimensionless quantity ξ is defined as the quantity in the square brackets. ξ is a product of two terms. The first is the ratio of maximum expander volume to the total (loading) volume of the SSR and is determined entirely by the geometry of the SSR. The second term is (to first order) the reciprocal of the molar concentration of ^3He loaded into the SSR. In the SSR's built to date, the values of ξ has typically varied from 1 to 100. Solutions for $T_e|_{\min}$ using Eqs. (51) and (54) are plotted in Fig. 11 in terms of ξ and for T_c values of 2.0, 1.8, 1.6, 1.4, 1.2, and 1.0 K.

Figure 11 is a quick reference to estimate the ultimate temperature of a superfluid Stirling machine given the overall geometry, the ^3He concentration loaded into the refrigerator, and the compressor operating temperature. For example, the SSR used in the experimental part of this work was loaded with a 3% mixture and has a maximum expander volume to loading volume ratio of 0.603 which results in a ξ value of 20.2. Using Fig. 11, we find that the ultimate operating temperature is limited to 1.72 K for $T_c = 2$ K, to 1.31 K for $T_c = 1.8$ K, to 0.68 K for $T_c = 1.6$ K, to 0.27 K for $T_c = 1.4$ K, to 0.09 K for $T_c = 1.2$ K, and to 0.027 K for $T_c = 1.0$ K. These ultimate temperatures are indicated in Figs. 5 through 8 by a dark arrow. A more conservative estimate can be made by including all the recuperator volume as part of the maximum expander volume. Then, the

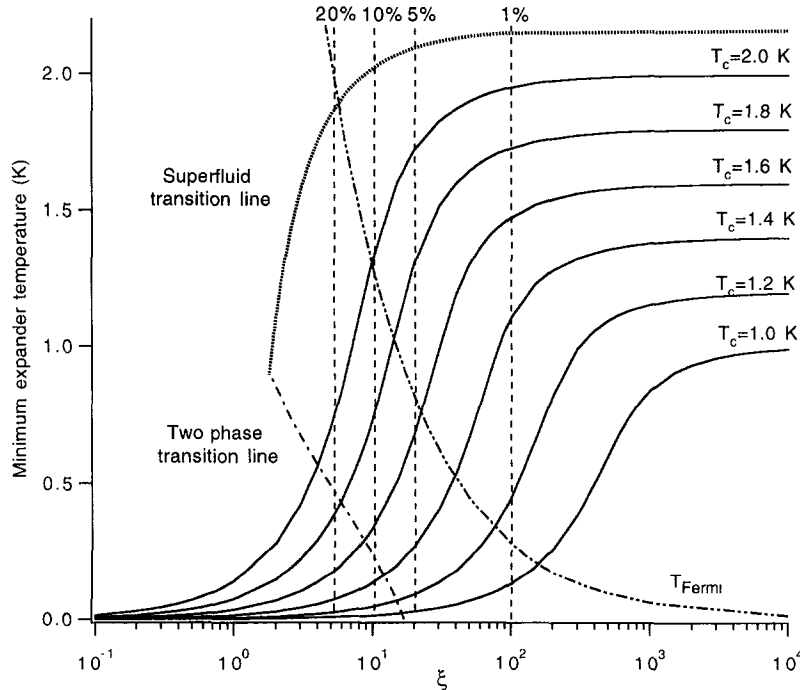


Fig. 11. A plot of the theoretically achievable expander temperature due to phonon-roton effects as a function of the dimensionless parameter ξ using Eq. (54) (solid lines). Each curve is labeled with the compressor operating temperature. The λ -line (dotted), the dilute phase line (dash-dot), and the ^3He Fermi temperature (dash-dot-dot) are also plotted on the same axes. The labeled vertical dashed lines indicate the mixture concentration in the expander for a given ultimate expander temperature and ξ .

maximum expander volume to loading volume ratio becomes 0.672 and ξ has a value of 22.6. Using Fig. 11, we find that the ultimate operating temperature is limited to 1.75 K for $T_c = 2$ K, 1.36 K for $T_c = 1.8$ K, to 0.76 K for $T_c = 1.6$ K, to 0.31 K for $T_c = 1.4$ K, to 0.10 K for $T_c = 1.2$ K, and to 0.03 K for $T_c = 1.0$ K. These ultimate temperatures are indicated in Figs. 5 through 8 by a light arrow. There is an apparent contradiction in Figs. 5 through 8 between the ultimate temperature calculated using Eq. (54) (Fig. 11) and the calculated zero cooling power using Eq. (45). In deriving Eq. (45) no limits were imposed on the sign of the pressure and number density of the ^3He . Under certain operating conditions, it is possible (in the PR model, not the experiment) to have a negative mass density of ^3He . This is clearly unphysical and the solutions of Eq. (54) determine the point when the PR model breaks down.

It becomes clear from Fig. 11 that high compressor temperatures necessitate operating an SSR with a small value of ξ to achieve low temperature. To achieve small values of ξ , the volume of the expander must be small fraction of the total volume of the SSR and the loading concentration of the mixture needs to be high.

In our SSR, the heat flush effects are largely irrelevant for determining the ultimate temperature for compressor temperatures between 1.2–1.6 K, as the low temperature performance of the SSR is dominated by the recuperator ineffectiveness. The heat flush effect is clearly seen in Fig. 8 where the compressor temperature is 1.8 K. The experimental curve is quite smooth until it achieves a temperature of 1.46 K where there is a definite “knee” in the cooling power curve as it drops abruptly to zero cooling power. Strictly speaking then, the PR model curves shown in Figs. 5–8 each should have knees where the cooling power abruptly drops to zero at the “arrowed” temperature. We believe that the difference between the knee temperature and the arrowed temperature in Fig. 8 can be attributed to the inaccuracies in the fountain pressure fit, in the values of the SSR volumes, and in the concentration of the mixture loaded into the SSR.

A consequence of a large recuperator volume is the broadening of the “heat flush transition” in the Q_c vs. T_e plot. As the expander temperature decreases, the region of pure superfluid ^4He begins to oscillate into and out of the recuperator, decreasing its effectiveness. If the recuperator has a large volume, there will be a large expander temperature range over which the thermal isolation afforded by the recuperator drops to zero.

The operation of the SSR is limited to the superfluid range of the phase diagram to allow superflow through the superleaks. The two-phase region is forbidden. As the SSR cools, the concentration of the ^3He increases in the expander. If the temperature is low enough and the concentration is high enough the mixture will spontaneously phase separate in the expander. This leads to a “locking” of the dilute ^3He concentration in the expander to the concentration associated with the phase transition at the expander temperature, precluding further cooling with the Stirling cycle.

Estimates on these limits can be accounted for in Fig. 11. ξ can be interpreted as a non-dimensional ^3He molar density at the presumed ultimate operating point. Using Eqs. (53) and (41), ξ can be written as

$$\xi = \frac{v_{3e}|_{\max}}{v_4^0} \quad (55)$$

where $v_{3e}|_{\max}$ is the molar volume of the ^3He when all the ^3He is placed into the maximum volume of the expander piston. ξ is then a comparison of the molar volume of the ^3He , when all of the ^3He in the SSR is in the

expander, to the molar volume of pure ^4He . Using Eq. (55), the superfluid transition for the mixture (the λ -line) and the dilute-phase two-phase boundary²¹ are also plotted in Fig. 11. SSR's designed with ξ 's to the left of these λ and two phase curves will have ultimate temperatures above that predicted by the solid curves in Fig. 11 because these (ξ, T) points are inaccessible to an operating SSR. The ultimate operating temperature of these machines can be determined by finding the maximum ^3He quasiparticle density in the expander as a function of expander temperature, using Eq. (26) and the ideal gas law. Where this curve crosses the dilute two phase line determines the estimate for the ultimate temperature of that design.

We have also plotted the Fermi temperature of the ^3He as a function of ξ in Fig. 11 as a double-dot single-dash curve. This curve is an indication of where the ^3He Fermi equation of state begins to deviate from that of an ideal Boltzmann gas and hence, marks where the PR model begins to break down. For our experimental SSR, the ultimate temperature estimates made earlier for compressor temperatures less than 1.4 K are rather suspect in that all the $\xi, T_e|_{\min}$ points fall well below and to the left of the Fermi temperature line. In this region, the Fermi gas model is more appropriate than the Boltzmann gas model for the ^3He quasiparticle gas. In generating the Fermi temperature curve we have assumed that the effective mass of the ^3He atoms in solution are 2.5 times the mass of a bare ^3He atom.²² Finally, we have included four vertical dashed lines, labeled 20%, 10%, 5% and 1% to indicate the ^3He concentration in the expander when an SSR (of a given ξ) is operated at its $T_e|_{\min}$.

Closing Comments

The performance limitations on the SSR described here are thermodynamic, imposed by the non-ideal thermodynamic properties of the working fluid. These limitations are not seen with the Schmidt model, which predicts an ultimate expander temperature of zero Kelvin. All assumptions concerning the transport properties have been ideal. For example, we have implicitly assumed that the viscosity is zero, that the thermal conductivity in the recuperator is infinite in the direction normal to the flow and zero in the direction parallel to the flow, and that the thermal contact in the piston cylinders to the thermal reservoirs is perfect. The overall performance of the real SSR will be worse than what is predicted by the PR model depending on the transport losses.

The PR model is simple and the equations of state that we have used above are not appropriate over the entire possible operating range of the SSR. At high concentrations, for example the ^3He does not behave entirely

as an ideal gas. In addition, the ^4He properties change as the ^3He concentration is increased. SSR's have been operated with ^3He concentrations as high as 36 %, ^{23,24} but, it is very questionable whether the ideal gas model is appropriate for these concentrations. Nevertheless, the PR model does very well in capturing many of the features and magnitudes of SSR performance for temperatures between 0.5 K and 2 K. This model is a useful tool to design an SSR and provides a useful handle on performance limitations.

Once again, we have outlined a new simple closed-form model for the high temperature operation of the superfluid Stirling refrigerator. This model is intended as a design tool to estimate the cooling power and ultimate temperature of any given design. The model shows that for temperatures above one Kelvin, the ^4He component significantly alters the performance of the SSR from that of standard Stirling machines. The National Science Foundation and the McCarthy chair supported this work.

REFERENCES

1. V. Kotsubo and G. W. Swift, "Superfluid Stirling Refrigerator: A New Method of Cooling Below 1 Kelvin," in *Proceedings of the 6th International Cryocoolers Conference*, David Taylor Research Center, Bethesda, MD (1991), pp. 59–70.
2. A. Watanabe, G. W. Swift, and J. G. Brisson, "Measurements with a Recuperative Superfluid Stirling Refrigerator," *Adv. Cryo. Eng.* **41**, 1527 (1996).
3. Ashok B. Patel and J. G. Brisson, to be published.
4. The original paper: Gustav Schmidt, "The theory of Lehmann's calorimetric machine," *Z. ver. Dtsch. Ing.* **15**, part 1 (1871). A more contemporary discussion can be found in: Israel Ureili and David M. Berchowitz, *Stirling Cycle Engine Analysis*, Adam Hilger Ltd., Bristol (1984).
5. J. G. Brisson and G. W. Swift, "High-Temperature Cooling Power of the Superfluid Stirling Refrigerator," *J. Low Temp. Phys.* **98**, 141 (1995).
6. J. G. Brisson and G. W. Swift, "Superfluid Stirling Refrigerator with a Counterflow Regenerator," in *Proceedings of the 7th International Cryocooler Conference*, Phillips Laboratory, Kirtland AFB, NM (1992), Part 2, p. 461.
7. Israel Ureili and David M. Berchowitz, *Stirling Cycle Engine Analysis*, Adam Hilger Ltd., Bristol (1984).
8. J. G. M. Kuerten, C. A. M. Castelijns, A. T. A. M. de Waele and H. M. Gijssman, "Thermodynamic Properties of Liquid ^3He - ^4He Mixtures at Zero Pressure for Temperatures below 250 mK and ^3He Concentrations Below 8%," *Cryogenics* **25**, pp. 419–442.
9. A. Th. A. M. de Waele and J. G. M. Kuerten, "Thermodynamics and Hydrodynamics of ^3He - ^4He Mixtures," in *Progress in Low Temperature Physics*, edited by D. F. Brewer, Elsevier, Amsterdam (1992), p. 174.
10. A. Watanabe, G. W. Swift, and J. G. Brisson, "Uniform Temperature Cooling Power of the Superfluid Stirling Refrigerator," *J. Low Temp. Phys.* **103**, 273 (1996).
11. C. Ebner and D. O. Edwards, "The Low Temperature Thermodynamic Properties of Superfluid Solutions of ^3He in ^4He ," *Phys. Reports* **2C**, 77 (1971).
12. A. Th. A. M. de Waele and J. G. M. Kuerten, "Thermodynamics and Hydrodynamics of ^3He - ^4He Mixtures," in *Progress in Low Temperature Physics*, edited by D. F. Brewer, Elsevier, Amsterdam (1992), p. 174.
13. See, for example, E. A. Guggenheim, *Thermodynamics, An Advanced Treatment for Chemists and Physicists*, 4th ed., North Holland, Amsterdam (1959), p. 213. Or, J. M.

- Smith and H. C. Van Ness, *Introduction to Chemical Engineering Thermodynamics*, 4th ed., McGraw-Hill, New York (1987), p. 424.
14. R. Radebaugh, "Thermodynamic Properties of ^3He - ^4He solutions with applications to the ^3He - ^4He dilution refrigerator," NBS Tech. Note 362 (1967), pp. 136-137.
 15. I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity*, W. A. Benjamin, Inc., New York (1965), Chap. 24.
 16. J. Wilks, *The Properties of Liquid and Solid Helium*, Clarendon, Oxford (1967), p. 666.
 17. I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity*, W. A. Benjamin, Inc., New York (1965), pp. 10-13.
 18. S. W. Van Sciver, *Helium Cryogenics*, Plenum Press, New York (1986), pp. 382-395.
 19. Ashok Patel and J. G. Brisson, "Experimental Performance of a Single Stage Superfluid Stirling Refrigerator," *J. Low Temp. Phys.* **111**, 201 (1998).
 20. Oxford Instruments Ltd., Oxford, England.
 21. R. De Bruyn Ouboter, K. Taconis, C. Le Pair, and J. Beenakker, "Thermodynamic Properties of Liquid ^3He - ^4He Mixtures Derived from Specific Heat Measurements Between 0.4 and 2 K over the Complete Concentration Range," *Physica* **25**, 853-888 (1960).
 22. R. Radebaugh, Thermodynamic properties of He^3 - He^4 solutions with applications to the He^3 - He^4 dilution refrigerator, Tech. Note 362, NBS (1967), pp. 7-10.
 23. A. Watanabe, G. W. Swift, and J. G. Brisson, "Uniform Temperature Cooling Power of the Superfluid Stirling Refrigerator," *J. Low Temp. Phys.* **103**, 273 (1996).
 24. A. Watanabe, G. W. Swift, and J. G. Brisson, "Measurements with a Recuperative Superfluid Stirling Refrigerator," *Adv. Cryo. Eng.* **41**, 1519 (1996).